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"Varopoulos paradigm": Mackey property versus metrizability in topological groups

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Abstract The class of all locally quasi-convex (lqc) abelian groups contains all locally convex vector spaces (lcs) considered as topological groups. Therefore it is natural to extend classical properties of locally convex spaces to this larger class of abelian topological groups. In the present paper we consider the following well known property of lcs: "A metrizable locally convex space carries its Mackey topology ". This claim cannot be extended to lqc-groups in the natural way, as we have recently proved with other coauthors (Außenhofer and de la Barrera Mayoral in J Pure Appl Algebra 216(6):1340–1347, 2012; Díaz Nieto and Martín Peinador in Descriptive Topology and Functional Analysis, Springer Proceedings in Mathematics and Statistics, Vol 80

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doi:10.1007/978-3-319-05224-3_7, 2014; Dikranjan et al. in Forum Math 26:723–757, 2014). We say that an abelian group G satisfies the *Varopoulos paradigm* (VP) if any metrizable locally quasi-convex topology on G is the Mackey topology. In the present paper we prove that in any unbounded group there exists a lqc metrizable topology that is not Mackey. This statement (Theorem C) allows us to show that the class of groups satisfying VP coincides with the class of finite exponent groups. Thus, a property of topological nature characterizes an algebraic feature of abelian groups.

Keywords Metrizable abelian groups · Locally quasi-convex topologies · Torsion groups · Precompact topologies · Locally convex spaces · Mackey topology

Mathematics Subject Classification Primary 22A05 · 43A40; Secondary 20E34 · 20K45

1 Introduction

1.1 Historical background and some facts on the Mackey topology

The locally convex topologies on linear spaces constitute a topic which has boosted a wide research in functional analysis. It was proved by Mackey [29] that the family of compatible locally convex topologies on a locally convex space has a top element. That is, for a (real or complex) locally convex space E with dual space E', there exists a locally convex topology $\mu = \mu(E, E')$ with $(E, \mu)' = E'$, such that any other locally convex topology ν on E such that $(E, \nu)' = E'$, satisfies the inequality $\nu \leq \mu$. The topology μ is named the *Mackey topology* of E. The following result will be the inspiring source for our search to extend the Mackey theory to abelian topological groups.

Theorem 1.1 [28, Corollary 22.3, page 210] *Metrizable locally convex linear topological spaces carry their Mackey topology.*

A parallelism between real topological vector spaces and abelian topological groups can be established by taking the circle group $\mathbb{T}$ as dualizing object for the class of abelian topological groups. The role of linear forms is played now by characters, and the dual group of a topological abelian group G is defined by $G^\wedge := CHom(G, \mathbb{T})$. A new group topology on a topological group G will be called compatible with the original one, if it gives rise to the same dual group $G^\wedge$. These basic concepts permit to study duality in the class of abelian topological groups and Varopoulos was the first to iniciate this task in [32].

The notion of quasi-convex subset of a topological abelian group (Vilenkin, [33]) paved the way to adapt the bulk of the Mackey theory from the class of topological vector spaces to the class of abelian topological groups. Seemingly, Varopoulos was unaware of the notion of locally quasi-convex group -introduced thirteen years earlier-when he wrote the magnificent paper [32]. Thus, he studied duality for the class of locally precompact groups, a subclass of that of locally quasi-convex groups. He proved in [32] that a metrizable locally precompact topological abelian group carries the finest compatible locally precompact group topology. In terms of the Definition 1.3, this could be stated as follows: A metrizable locally precompact group is Mackey in the class of locally precompact groups (or $\mathcal{L}pc$-Mackey). The mentioned result of Varopoulos originated the title and the contents of the present paper: in fact, we study it in the class of locally quasi-convex groups.

The Mackey theory in the framework of locally quasi-convex groups was considered for the first time in [11], where a rather complete outlook of it is offered. It can be seen there that the techniques of locally convex spaces cannot be simply translated to this new setting, and convenient tools must be provided to obtain counterparts of the main results for locally convex spaces. Although a topological vector space is locally convex if and only if it is locally quasi-convex as a topological group [5, 2.4], there are some obstructions to extend the Mackey–Arens theorem to the class of all locally quasi-convex groups, so the Mackey problem is stated as follows in [11]:

Problem 1.2 Let (G, τ) be a locally quasi-convex topological group. Let $\mathcal{C}(G, \tau)$ denote the family of all locally quasi-convex group topologies on G which are compatible with τ.

Does there exist a top element in the family $\mathcal{C}(G, \tau)$?

If it exists, we denote it by $\mu = \mu(G, \tau)$. Then (G, μ) will be called a Mackey group and μ the Mackey topology for (G, τ). Since the family $\mathcal{C}(G, \tau)$ is fixed by the pair $(G, G^\wedge)$, the topology μ can be also referred to as the Mackey topology in the duality $(G, G^\wedge)$.

In the class of locally quasi-convex groups there always exists a bottom element in $\mathcal{C}(G, \tau)$, namely the weak topology on G induced by $(G, \tau)^\wedge$. Thus, if a precompact topology is Mackey (cf. Theorem B (ii)), then it is the unique compatible topology.

This problem can be generalized to other classes of abelian topological groups. To this end we give the following:

Definition 1.3 Let $\mathcal{G}$ be a class of abelian topological groups and let (G, τ) be a topological group in $\mathcal{G}$. Let $\mathcal{C}_\mathcal{G}(G, \tau)$ denote the family of all $\mathcal{G}$-topologies ν on G compatible with τ. We say that $\mu \in \mathcal{C}_\mathcal{G}(G, \tau)$ is the *$\mathcal{G}$-Mackey topology* for $(G, G^\wedge)$ (or the *Mackey topology for* (G, τ) *in* $\mathcal{G}$) if $\nu \leq \mu$ for all $\nu \in \mathcal{C}_\mathcal{G}(G, \tau)$.

If $\mathcal{G}$ is the class of locally quasi-convex groups $\mathcal{L}qc$, we will simply say that the topology is Mackey.

Remark 1.4 The Mackey topology exists in $\mathcal{L}qc$ if and only if the supremum of two compatible topologies in $\mathcal{L}qc$ is again compatible ([11, Proposition 3.11]).

Problem 1.5 Let $\mathcal{G}$ be a class of topological abelian groups. Does there exist a top element in the family $\mathcal{C}_\mathcal{G}(G, \tau)$, i.e., does the $\mathcal{G}$-Mackey topology exist?

According to Barr and Kleisli [7], the $\mathcal{N}$-Mackey topology exists in the class $\mathcal{N}$ of all nuclear groups. The latter class was introduced by Banaszczyk in [5] and contains all locally compact abelian groups and all nuclear vector spaces.

The problem whether the Mackey topology exists, in the setting of $\mathcal{L}qc$-groups, is so far open. Certain conditions of topological nature imposed on the group guarantee that its own topology is already the Mackey topology. For instance, this happens in the following classes of topological groups:

Theorem 1.6 [11] *Let* (G, τ) *be a locally quasi-convex topological group satisfying one of the following conditions:*

1. *G is Baire and separable,*
2. *G is Čech-complete (in particular, G is metrizable and complete),*
3. *G is pseudocompact.*

Then τ *is the Mackey topology in the duality* $(G, G^\wedge)$.

The fact that every metrizable and complete locally quasi-convex group is Mackey, points out an analogy between topological groups and topological vector spaces where the counterpart of this result is available (see Theorem 1.1). On the other hand, there are metrizable locally quasi-convex groups which are not Mackey. Thus, completeness seems to be an essential requirement.

We present below a list of examples witnessing that these properties might combine in different forms. First recall that a group topology is called *linear*, if it has a local base at 0 consisting of subgroups (see Definition 1.18).

Example 1.7 The following groups admit a metrizable non-discrete locally quasi-convex topology which is not Mackey. Since they are countable they cannot be complete, for otherwise, by the Baire Category Theorem they would be discrete.

(a) The group of the integers $\mathbb{Z}$ endowed with any non-discrete linear topology is metrizable and locally quasi-convex but fails to be Mackey [1].
(b) The Prüfer groups $\mathbb{Z}(p^\infty)$ endowed with the topology inherited from $\mathbb{T}$ are metrizable locally-quasi-convex groups which are not Mackey [8].
(c) The group of rationals $\mathbb{Q}$ endowed with the topology inherited from the real line is a *non precompact* metrizable locally quasi-convex group which is not Mackey [8,16].

Example 1.8 The following groups are metrizable locally quasi-convex and Mackey:

Let $m \geq 2$ be a natural number and consider the direct sum of countably many copies of $\mathbb{Z}_m$, $G_m := \bigoplus_\omega \mathbb{Z}_m$ endowed with the topology inherited from the product $\prod_\omega \mathbb{Z}_m$. Then G_m is a metrizable, locally quasi-convex *non-complete* Mackey group [10].

One can also find examples with a stronger flavor from Functional Analysis, namely connected, metrizable topological groups which are not Mackey. Indeed, the group of all the null-sequences on $\mathbb{T}$, endowed with the topology inherited from the product $\mathbb{T}^\mathbb{N}$ is a metrizable connected precompact locally quasi-convex group which is not Mackey [21]. Relevant properties of this group endowed with other topologies can be seen in [26].

A nice survey about the development of the duality theory for groups from [32] to other classes of groups can be found in [30].

1.2 Main results

Metrizability alone does not ensure that a locally quasi-convex topology on a group is Mackey, unless it is accompanied (or replaced) by an additional topological property (completeness, etc., see Theorem 1.6). This motivates to investigate additional properties of *algebraic* nature, that interacting with metrizability guarantee that a locally quasi-convex topological group is a Mackey group.

Definition 1.9 We say that an abelian group G satisfies the *Varopoulos Paradigm*, if every metrizable locally quasi-convex topology on G must be a Mackey topology. We briefly denote this by $G \in \mathbf{VP}$.

According to the main theorem of this paper, the groups satisfying the Varopoulos Paradigm are exactly the bounded ones:

Theorem A: *Let G be an abelian group. Then, the following assertions are equivalent:*

(i) $G \in \mathbf{VP}$, *i.e., every metrizable locally quasi-convex group topology on G is Mackey.*
(ii) G is bounded.

For the proof of Theorem A, we will consider separately the cases of bounded and unbounded groups. In each case we will get one of the implications of Theorem A. In fact, Theorem B is a stronger version of the implication $(ii) \Rightarrow (i)$ in Theorem A, while Theorem C is simply the implication $(i) \Rightarrow (ii)$.

Here comes our main result in the case of bounded groups. It provides two sufficient conditions for a locally quasi-convex topology to be a Mackey topology.

Theorem B: *Let G be a bounded abelian group and* τ *a locally quasi-convex topology on* G.

(i) If τ *is metrizable, then it is the Mackey topology for* G.
(ii) If $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$, *then* τ *is precompact and the Mackey topology for* G, *or equivalently* $|\mathcal{C}(G,\tau)| = 1$.

Immediately from Theorem B (ii) we get:

Corollary 1.10 *Let* (G,τ) *be a locally quasi-convex non-precompact group topology on a bounded group. Then* $\left|(G,\tau)^\wedge\right| \geq \mathfrak{c}$.

Remark 1.11 In [10, Example 4] a weaker version of Theorem B (ii) was shown. Namely: every bounded precompact topological group (G,τ) with $\left|(G,\tau)^\wedge\right| < \mathfrak{c}$ is Mackey.

In [14, Prop. 8.57] Theorem B (ii) was proved by other methods.

Example 1.12 The following examples show that none of the hypotheses of Theorem B can be omitted. Here and frequently in the sequel, we identify $\mathbb{T}$ with the quotient group $\mathbb{R}/\mathbb{Z}$ in order to have additive notation available.

(a) The p-adic topology on $\mathbb{Z}$ is locally quasi–convex (even linear), metrizable, and precompact, but it is not Mackey [1]. Its dual group is countable. Thus boundedness is essential.

(b) Consider a D-sequence $\mathbf{b} = (b_n)$ satisfying $\frac{b_{n+1}}{b_n} \to \infty$ as in [1]. Let $\tau_{\mathbf{b}}$ be the topology on $\mathbb{Z}$ of uniform convergence on the set $\left\{\frac{1}{b_n} + \mathbb{Z}\right\} \subset \mathbb{T}$. As proved in [1], $\tau_{\mathbf{b}}$ is metrizable and locally quasi-convex and satisfies $\left|(\mathbb{Z}, \tau_{\mathbf{b}})^\wedge\right| < \mathfrak{c}$ but it is not precompact.
(c) The Prüfer group $\mathbb{Z}(p^\infty)$ endowed with the topology induced from $\mathbb{T}$ is a precompact, metrizable torsion group with countable dual group which is not Mackey according to [8]. This shows that we cannot replace "bounded" by "torsion groups".
(d) The direct sum of $\mathfrak{c}$-many copies of $\mathbb{Z}_m$ endowed with the topology inherited from the product $\mathbb{Z}_m^{\mathfrak{c}}$ is precompact, bounded (therefore linear) and its dual group has cardinality $\mathfrak{c}$. It will be proved in the forthcoming paper [2] that it is Mackey. This shows that the first implication of Theorem B (ii) cannot be inverted.
(e) For any natural $m > 1$ the group $G = \bigoplus_\omega \mathbb{Z}_m$ endowed with the discrete topology is bounded, locally quasi-convex with $\left|G^\wedge\right| = \mathfrak{c}$, but it is not precompact.
(f) Let G be an infinite (bounded) group. Let δ^+ denote the Bohr topology on G, i.e. the weak topology on G induced by all homomorphisms $G \to \mathbb{T}$. Then (G, δ^+) is (bounded) precompact, $w(G, \delta^+) = 2^{|G|} \geq \mathfrak{c}$ and it is not Mackey, as the discrete topology is the Mackey topology in the dual pair $(G, (G, \delta^+)^\wedge)$.

From Theorem B (and some extra results from [3]), we compute the cardinality of the family $\mathcal{C}(G, \tau)$ for a metrizable bounded group, as expressed next:

Corollary B1: *Let (G, τ) be a metrizable, locally quasi-convex bounded group. Then:*

1. $|\mathcal{C}(G, \tau)| = 1$ *if and only if τ is precompact. In this case, (G, τ) is a precompact Mackey group.*
2. *Otherwise $|\mathcal{C}(G, \tau)| \geq 2^{\mathfrak{c}}$ and τ is the only metrizable topology in $\mathcal{C}(G, \tau)$.*

Corollary B1 solves Conjecture 7.6 in [3] for bounded groups.

Remark 1.11 and Theorem B (ii) suggest to pay special attention to precompact topologies on bounded abelian groups. Since $\left|(G, \tau)^\wedge\right| = w(G, \tau)$ for a precompact topology τ, it will be convenient to use the following notation from [9] (appropriately modified later in [17,18]). For an infinite group G and an infinite cardinal κ, we denote by $\mathcal{B}_\kappa(G)$ the family of all precompact topologies of weight κ on G. Then, $\mathcal{B}_\kappa(G) \neq \emptyset$ only when $\kappa \leq 2^{|G|}$ and $|G| \leq 2^\kappa$; in such a case $|\mathcal{B}_\kappa(G)| = 2^\kappa$ ([9, Theorem 5.3], see also [31]). In particular, there exist $2^{2^{|G|}}$ precompact topologies of weight $2^{|G|}$ on G.

In these terms, $\mathcal{B}_\kappa(G)$ entirely consists of Mackey topologies when G is bounded and $\kappa < \mathfrak{c}$. Things become especially neat in the case of countable groups considered in the remark below.

Remark 1.13 Let G be a countably infinite group and κ an infinite cardinal. As mentioned above, there exist $2^{\mathfrak{c}}$ precompact topologies of weight $\mathfrak{c}$ on G. As far as infinite weights $\kappa < \mathfrak{c}$ are concerned, one may have $|\mathcal{B}_\kappa(G)| = 2^\kappa > \mathfrak{c}$ in general (e.g., under the assumption that both the Continuum Hypothesis and the Lusin's Hypothesis $[2^{\omega_1} = \mathfrak{c}]$ fail, one has $|\mathcal{B}_{\omega_1}(G)| = 2^{\omega_1} > \mathfrak{c}$ while $\omega_1 < \mathfrak{c}$). On the other hand, under Martin Axiom, $2^\kappa = \mathfrak{c}$ for all κ with $\omega \leq \kappa < \mathfrak{c}$. Hence, under Martin Axiom,

$$\left| \bigcup_{\kappa<\mathfrak{c}} \mathcal{B}_\kappa(G) \right| = \mathfrak{c}, \tag{1}$$

i.e., there exist precisely $\mathfrak{c}$ precompact topologies of weight $< \mathfrak{c}$ on G. The equality (1) shows that under the assumption of Martin Axiom the family of precompact topologies of the maximum weight $\mathfrak{c}$ strongly prevails over the rest of precompact topologies of G.

For a bounded abelian group, the case of precompact topologies of weight $< \mathfrak{c}$ is completely resolved by Theorem B (ii), namely *all of them are Mackey*. In particular, a non-Mackey precompact topology on a countable bounded abelian group has necessarily weight $\mathfrak{c}$. This is why we pass now to the case of precompact topologies of weight $\mathfrak{c}$. Here things change completely, and one has the maximum number ($2^{\mathfrak{c}}$) of non-Mackey topologies (necessarily of weight $\mathfrak{c}$):

Proposition 1.14 *Let G be a bounded countably infinite group. Then, $\mathcal{B}_{\mathfrak{c}}(G)$ contains $2^{\mathfrak{c}}$ non-Mackey topologies.*

Proof Since G is bounded and countably infinite, there exists a natural number L such that $G = \bigoplus_{n<\omega} \mathbb{Z}_{m_n}$, where $m_n < L$ for all $n \in \mathbb{N}$. Hence, we can write $G = G_1 \times G_2$ where both G_1 and G_2 are infinite. Since G_1 is infinite, there exists a family of $2^{\mathfrak{c}}$ precompact topologies $\{\tau_i\}$ on G_1 of weight $\mathfrak{c}$ by Remark 1.13. Consider the discrete topology δ in G_2 and in G the product topology, namely $\mathcal{T}_i = \tau_i \times \delta$. For each i the topology $\mathcal{T}_i$ is locally quasi convex but not precompact, i.e., $\mathcal{T}_i > \mathcal{T}_i^+ = \tau_i \times \delta^+$. In addition, $\mathfrak{c} \geq w(G, \mathcal{T}_i^+) \geq w(G_2, \delta^+) = \mathfrak{c}$. Hence the topologies $\mathcal{T}_i^+$ are precompact of weight $\mathfrak{c}$ and not Mackey. $\square$

This proposition leaves the following natural question:

Question 1.15 Let G be a countable bounded abelian group. How many Mackey topologies does $\mathcal{B}_{\mathfrak{c}}(G)$ contain?

We do not know the answer even of the more specific question of existence Mackey topologies of weight $\mathfrak{c}$:

Question 1.16 For a countable bounded group G does $\mathcal{B}_{\mathfrak{c}}(G)$ contain any Mackey topologies ?

In Sect. 2, we prove Theorem B and Corollary B1.

Remark 1.17 We shall show in another paper that in the class of linearly topologized groups the Mackey topology exists and that every metrizable linear topology is the Mackey topology in this class. Since the notions Mackey and $\mathcal{L}in$-Mackey coincide for bounded groups, the announced result solves the Mackey Problem in the setting of bounded groups.

For unbounded groups, the picture is completely different. In fact, on any unbounded topological group we can build a metrizable locally quasi-convex group topology which is not Mackey. This is Theorem C, which is the main result in this direction:

Theorem C: *If G is an unbounded group, then $G \notin$ **VP**, i.e., there exists a metrizable locally quasi-convex topology on G which is not Mackey.*

The proof of Theorem C, given in Sect. 3, is based in the following facts:

- $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$ and $\mathbb{Q}$ have a metrizable locally quasi-convex non-Mackey topology (see Example 1.7).
- Every unbounded group G has a subgroup of the form $\mathbb{Z}$, $\mathbb{Z}(p^\infty)$, $\mathbb{Q}$ or $\bigoplus_{n=1}^\infty \mathbb{Z}_{m_n}$, where p is a prime and (m_n) is a sequence with $\lim m_n = \infty$ [25].
- Open subgroups of Mackey groups are Mackey (Lemma 3.2).
- For any sequence of natural numbers (m_n) with $m_n \to \infty$ the product topology of the group $\bigoplus_{n=1}^\infty \mathbb{Z}_{m_n}$ is metrizable and non-Mackey (Proposition 3.6), witnessing $\bigoplus_{n=1}^\infty \mathbb{Z}_{m_n}$ $\notin$ **VP**.

As a by-product, the construction in Proposition 3.6 (of a compatible locally quasi-convex topology finer than the product topology) can be used to provide $\mathfrak{c}$ many pairwise non-isomorphic metrizable locally precompact linear group topologies on the group $\mathbb{Q}/\mathbb{Z}$ and all its subgroups of infinite rank (compare with Example 1.7(c), where the single topology with similar properties is not linear).

1.3 Notation and terminology

The symbols $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}(p^\infty), \mathbb{T}$ will stand for the natural numbers, the integer numbers, the rational numbers, the complex numbers, the Prüfer groups and the unit circle of the complex plane respectively. Using the structure of $\mathbb{C}$, in particular, the real part function $Re(z)$ for $z \in \mathbb{C}$, we let $\mathbb{T}_+ = \mathbb{T} \cap \{z \in \mathbb{C} : Re(z) \geq 0\}$ (although we frequently identify $\mathbb{T}$ with the quotient group $\mathbb{R}/\mathbb{Z}$, if the additive notation is more convenient). The group $\mathbb{Z}_m$ denotes the cyclic group of m elements. For an abelian group G, we denote by G_d the group endowed with its discrete topology and $G^* = G_d^\wedge = \mathrm{Hom}(G, \mathbb{T})$. For a cardinal κ we denote by $G^{(\kappa)}$ the direct sum $\bigoplus_\kappa G$ of κ copies of G.

For a topological group (G, τ) its dual group is the group of continuous homomorphisms $\mathrm{CHom}(G, \mathbb{T})$, and will be denoted by $(G, \tau)^\wedge$ or $G^\wedge$ if the topology is clear. Two group topologies τ and ν on an abelian group G are said to be compatible if $(G, \tau)^\wedge = (G, \nu)^\wedge$. For a topological group (G, τ), we denote by $\mathcal{C}(G, \tau)$ the family of all locally quasi-convex topologies on G which are compatible with τ. A subgroup $H \leq G$ of a topological group is dually closed if for any $x \notin H$, there exists $\phi \in G^\wedge$ satisfying that $\phi(H) = 1$ and $\phi(x) \neq 1$.

For an abelian group G and a subgroup $H \leq \mathrm{Hom}(G, \mathbb{T})$ we shall denote by $\sigma(G, H)$ the weak topology on G induced by the elements of H. When the initial group is a topological group, then $G^\wedge \leq \mathrm{Hom}(G, \mathbb{T})$ is a subgroup and we write τ^+ instead of $\sigma(G, G^\wedge)$. The topology τ^+ is usually called the Bohr topology of (G, τ). Obviously, H is dually closed if it is τ^+-closed. If G carries the discrete topology, the corresponding Bohr topology has remarkable properties (see for example [23]).

A group G is said to be bounded if there exists $m \in \mathbb{N}$ satisfying that $mG = 0$. In this case, we will call the exponent of G to the minimum positive integer satisfying this condition. For a prime number p, we say that a group G is a p-group if for every $x \in G$ there exists a natural number n satisfying that $p^n x = 0$.

Definition 1.18 Let (G, τ) be a topological group. We say that τ is *linear* if it has a neighborhood basis consisting in subgroups. In this case, we say that (G, τ) is *linearly topologized*.

Definition 1.19 Let (G, τ) be a topological group. We say that a subset $M \subset G$ is *quasi-convex* if for every $x \in G \backslash M$, there exists $\phi \in G^\wedge$ satisfying that $\phi(M) \subset \mathbb{T}_+$ and $\phi(x) \notin \mathbb{T}_+$. If τ has a neighborhood basis consisting in quasi-convex sets, we say that τ is *locally quasi-convex*.

For subsets $M \subseteq G$ and $N \subseteq G^\wedge$, we define $M^\triangleright := \{\chi \in G^\wedge : \ \chi(M) \subseteq \mathbb{T}_+\}$ and $N^\triangleleft := \{x \in G : \ \chi(x) \in \mathbb{T}_+ \ \forall \chi \in N\}$. They are called the polar and the inverse polar of M and N, respectively.

For the reader's convenience, we include a proof of the following result, which can be also found in [4, Proposition 2.1].

Proposition 1.20 *Let (G, τ) be a locally quasi-convex bounded group. Then τ is linear.*

Proof Suppose that $mG = 0$. Let U be a quasi-convex neighborhood of 0 in G. It is well known that $U^\triangleright$ is an equicontinuous subset of $G^\wedge$. Hence there exists a neighborhood W of 0 in G such that $\chi(W) \subseteq]-\frac{1}{m}, \frac{1}{m}[+\mathbb{Z}$ for all $\chi \in U^\triangleright$. Since $m\chi \equiv 0$, we obtain $\chi(W) \subseteq \chi(G) \subseteq \{\frac{k}{m} + \mathbb{Z}, \ k \in \mathbb{Z}\}$. Combined with the above inclusion, this yields $\chi(W) = \{0\}$ for every $\chi \in U^\triangleright$. This means $W \subseteq (U^\triangleright)^\perp$. In particular, the subgroup $(U^\triangleright)^\perp \subseteq (U^\triangleright)^\triangleleft = U$ is open. □

Corollary 1.21 *The notions $\mathcal{L}in$-Mackey and Mackey are equivalent for bounded groups.*

We will consider the following notation for some classes of topological groups:

- $\mathcal{L}in$—for the class of linearly topologized Hausdorff groups.
- $\mathcal{L}pc$—for the class of locally precompact groups.
- $\mathcal{L}qc$—for the class of locally quasi convex Hausdorff groups.

The terms $\mathcal{L}in$-Mackey, $\mathcal{L}pc$-Mackey, $\mathcal{L}qc$-Mackey have the meaning established in Definition 1.3.

2 The bounded case: proofs of Theorem B and Corollary B1

Proposition 2.1 *Let G be an abelian group. If τ, τ' are compatible topologies on G, they share the same dually closed subgroups.*

Proof The assertion trivially follows from these two facts:

(a) both topologies have the same weak topology $\sigma(G, G^\wedge)$;
(b) a subgroup H of G is dually closed (with respect to any of these two topologies) precisely when it is $\sigma(G, G^\wedge)$-closed. □

Next, we want to show that a metrizable locally quasi-convex bounded group is Mackey. For the proof we need the following Proposition which is interesting on its own. Although it follows easily from a stronger property (see Remark 2.3(a)), we give a self contained accessible proof of it.

Proposition 2.2 *Let (G, τ) be a torsion, metrizable non-discrete abelian group. Then there exists a homomorphism $f : G \to \mathbb{T}$ which is not continuous.*

Proof Since G is metrizable and τ is not the discrete topology, we can choose a null-sequence $(a_n)_{n\in\mathbb{N}}$ which satisfies $a_n \neq a_m$ whenever $n \neq m$ and $a_n \neq 0$ for all $n \in \mathbb{N}$. Since every finitely generated torsion group is finite we may additionally assume that a_{n+1} does not belong to the subgroup generated by $a_1, \ldots, a_n$ for every $n \in \mathbb{N}$.

We are constructing inductively a homomorphism $f_n : \langle a_1, \ldots, a_n\rangle \to \mathbb{T}$ in the following way:

Let $f_1 : \langle a_1\rangle \to \mathbb{T}$ be a homomorphism which satisfies $f_1(a_1) \notin \mathbb{T}_+$. This is possible, since $a_1 \neq 0$.

Assume $f_n : \langle a_1, \ldots, a_n\rangle \to \mathbb{T}$ is a homomorphism, which satisfies $f_n(a_k) \notin \mathbb{T}_+$ for all $1 \leq k \leq n$.

If $\langle a_{n+1}\rangle \cap \langle a_1, \ldots, a_n\rangle = \{0\}$, a homomorphism $f_{n+1} : \langle a_1, \ldots, a_{n+1}\rangle \to \mathbb{T}$ which extends f_n and satisfies $f_{n+1}(a_{n+1}) \notin \mathbb{T}_+$ can be defined in an obvious way.

If $\langle a_{n+1}\rangle \cap \langle a_1, \ldots, a_n\rangle \neq \{0\}$, we choose the minimal natural number $k \in \mathbb{N}$ such that $ka_{n+1} = k_1a_1 + \cdots + k_na_n$ for suitable $k_1, \ldots, k_n \in \mathbb{Z}$. By assumption, $2 \leq k < \operatorname{ord}(a_{n+1})$. We define $f_{n+1}(a_{n+1})$ such that $f_{n+1}(a_{n+1}) \notin \mathbb{T}_+$ and $f_{n+1}(ka_{n+1}) = f_n(k_1a_1 + \cdots + k_na_n)$ and obtain a homomorphism $f_{n+1} : \langle a_1, \ldots, a_{n+1}\rangle \to \mathbb{T}$ which extends f_n.

In this way, we obtain a homomorphism $\widetilde{f} : \langle\{a_n : n \in \mathbb{N}\}\rangle \to \mathbb{T}$ by letting $\widetilde{f}(a_k) = f_n(a_k)$, for some (i.e. all) $n \geq k$. The homomorphism $\widetilde{f}$ can be extended to a homomorphism $f : G \to \mathbb{T}$. By construction, f satisfies:

$$\forall n \in \mathbb{N}, \;\; f(a_n) \notin \mathbb{T}_+.$$

So f is not continuous. □

Remark 2.3 (a) Using the fact that no non-trivial sequence can converge in the Bohr topology δ_G^+ of the discrete group G [24], one can obtain a proof of a stronger version of the above proposition with "torsion" omitted. (Just take any non-trivial null sequence (a_n) and note that it cannot converge in δ_G^+, hence there exists a character χ of G witnessing that, i.e., such that $\chi(a_n) \not\to 0$ in $\mathbb{T}$; then χ cannot be τ-continuous, as $a_n \to 0$ in (G, τ).)

(b) If we reinforce the hypothesis of the above proposition replacing "torsion" by "bounded", the proof can be simplified in the following way. Passing to a subsequence of (a_n) one can achieve to have $\langle a_{n+1}\rangle \cap \langle a_1, \ldots, a_n\rangle = \{0\}$ to avoid the more complicated second case ($\langle a_{n+1}\rangle \cap \langle a_1, \ldots, a_n\rangle \neq \{0\}$) in the above proof.

Proof of Theorem B: Let τ be a locally quasi-convex topology on a bounded group G. We have to prove that (G, τ) is a Mackey group provided that: (i) τ is metrizable; or (ii) $\left|(G, \tau)^\wedge\right| < \mathfrak{c}$.

(i) Suppose that τ is metrizable. Let τ' be another locally quasi-convex group topology on G compatible with τ. According to 1.20, τ and τ' are linear topologies. We have to show that $\tau' \subseteq \tau$. So fix a τ'-open subgroup H of G and denote by $q : G \to G/H$ the canonical projection. As H is open in (G, τ'), it is also dually closed in (G, τ') and according to 2.1 also in (G, τ). Let $f : G/H \to \mathbb{T}$ be an arbitrary homomorphism. Since the restriction of $f \circ q$ to the τ'-open subgroup H is null, we have $f \circ q \in (G, \tau')^\wedge = (G, \tau)^\wedge$. Let $\hat{\tau}$ denote the Hausdorff quotient topology on G/H induced by τ. Of course, $(G/H, \hat{\tau})$ is a metrizable bounded group. Since every homomorphism $f : (G/H, \hat{\tau}) \to \mathbb{T}$ is continuous, we obtain that G/H is discrete by the above Proposition. This implies that H is τ-open. Hence, $\tau' \subseteq \tau$.

(ii) Now, suppose that $\left|(G, \tau)^\wedge\right| < \mathfrak{c}$. By Proposition 1.20, τ is linear. Suppose that τ is not precompact. Then there exists an open subgroup $U \leq G$ satisfying that $|G/U| \geq \omega$. Since U is open, the quotient G/U is discrete. Hence $(G/U)^\wedge$ is compact and infinite, therefore $\left|(G/U)^\wedge\right| \geq \mathfrak{c}$. Since $\left|G^\wedge\right| \geq \left|(G/U)^\wedge\right|$ the inequality $\left|(G, \tau)^\wedge\right| \geq \mathfrak{c}$ follows. This contradiction shows that τ is precompact.

Let τ' be another locally quasi-convex group topology on G, which is compatible with τ. As $(G, \tau')^\wedge = (G, \tau)^\wedge$, we conclude that $\left|(G, \tau')^\wedge\right| < \mathfrak{c}$. So, the above argument applied to τ' shows that τ' is precompact as well. Since there is exactly one precompact topology in a duality, τ and τ' must coincide. Therefore, τ is Mackey.

Proof of Corollary B1: We have to prove that if (G, τ) is a metrizable, locally quasi-convex and bounded group, then:

1. $|\mathcal{C}(G, \tau)| = 1$ if and only if τ is precompact.
2. otherwise $|\mathcal{C}(G, \tau)| \geq 2^{\mathfrak{c}}$; in this case, τ is the only metrizable topology in $\mathcal{C}(G, \tau)$.

If $|\mathcal{C}(G, \tau)| = 1$ it is clear that τ is precompact (for any dual pair $(G, G^\wedge)$ there exists one precompact topology). Conversely, if τ is precompact, Theorem B (i) implies that τ is Mackey. Hence it is clear that $|\mathcal{C}(G, \tau)| = 1$.

The proof of 2. relies on the fact that τ is linear (Proposition 1.20). Assume that (G, τ) is not precompact. Then, there exists a τ-open subgroup U such that G/U is infinite. Since U is open, G/U is discrete. By Theorem 3.10 in [3], the family $\mathcal{C}(G/U)$ can be embedded in $\mathcal{C}(G, \tau)$. We denote by $\mathcal{F}il_{G/U}$ the set of all filters on G/U and we recall that $\left|\mathcal{F}il_{G/U}\right| = 2^{2^{|G/U|}}$. By Theorem 4.5 in [3], we have the equality $|\mathcal{C}(G/U)| = \left|\mathcal{F}il_{G/U}\right|$, since G/U has infinite rank (being an infinite bounded group). So

$$|\mathcal{C}(G, \tau)| \geq |\mathcal{C}(G/U)| = 2^{2^{|G/U|}} \geq 2^{2^{\omega}} = 2^{\mathfrak{c}}$$

holds. It is also clear from Theorem B, that τ is the only metrizable topology in $\mathcal{C}(G, \tau)$, which coincides with the top element in $\mathcal{C}(G, \tau)$.

3 The unbounded case: proof of Theorem C

In order to prove that unbounded groups do not fit into Varopoulos paradigm, we need first to prove that the Mackey topology is "hereditary" for open subgroups. In other words, an open subgroup of a Mackey group is also a Mackey group in the induced topology. To this end we recall next an standard extension of a topology from a subgroup to the whole group.

Definition 3.1 Let G be an abelian group and H a subgroup. Consider on H a group topology τ. The *extension* $\overline{\tau}$ of τ is the topology on G which has as a basis of neighborhoods at zero the neighborhoods at zero of (H, τ).

If the topology τ on H is metrizable, then the topology $\overline{\tau}$ on G is also metrizable. Analogously, if τ is locally quasi-convex, then $\overline{\tau}$ is also locally quasi-convex (indeed, a subset which is quasi-convex in H is quasi-convex in G, due to the fact that H is an open subgroup in $\overline{\tau}$).

Proposition 3.2 *Let H be an open subgroup of the topological group (G, τ). If τ is the Mackey topology, then the induced topology $\tau_{|H}$ is also Mackey.*

Proof In order to prove that $(H, \tau_{|H})$ is also a Mackey group take a locally quasi-convex topology ν in H such that $(H, \nu)^\wedge = (H, \tau_{|H})^\wedge$. Let $\overline{\nu}$ be the extension of ν to G.

Let us prove that $(G, \overline{\nu})^\wedge = (G, \tau)^\wedge$. Take $\phi \in (G, \overline{\nu})^\wedge$. Then $\phi_{|H}$ is ν-continuous, and, since ν and $\tau_{|H}$ are compatible, $\phi_{|H}$ is also a character of $(H, \tau_{|H})$. Since H is τ-open, ϕ is τ-continuous. Thus $(G, \overline{\nu})^\wedge \leq (G, \tau)^\wedge$. The same argument can be used to prove the converse inequality, taking into account that H is an open subgroup in $\overline{\nu}$. So $\overline{\nu}$ and τ are compatible topologies.

Since (G, τ) is Mackey, we obtain that $\overline{\nu} \leq \tau$ and this inequality also holds for their restrictions to H, $\nu \leq \tau_{|H}$. Hence $(H, \tau_{|H})$ is Mackey. □

For later use we state the following:

Corollary 3.3 *If ν is a metrizable, locally quasi-convex* non-Mackey *group topology on a subgroup H of a group G, then H, then there exists a metrizable locally quasi-convex-group topology on G which is not Mackey, namely $\overline{\nu}$.*

Proof Let ν be a metrizable locally quasi-convex group topology on H which is not Mackey and $\overline{\nu}$ its extension to G. Clearly, H is $\overline{\nu}$-open, and $\overline{\nu}$ is a metrizable locally quasi-convex topology on G. If $\overline{\nu}$ were Mackey, by our previous arguments, ν should be Mackey as well, a contradiction. □

We do not know if the converse of Proposition 3.2 holds in general:

Question 3.4 Let (G, τ) be a topological group and $H \leq G$ an open subgroup. If $\tau_{|H}$ is the Mackey topology, is τ the Mackey topology for G?

We prove it for the class of bounded groups.

Proposition 3.5 *Let (G, τ) be a bounded topological abelian group and $H \leq G$ an open subgroup. If $(H, \tau_{|H})$ is Mackey, then (G, τ) is Mackey.*

Proof Let ν be a locally quasi-convex group topology on G which is compatible with τ. Note first that τ, ν, $\tau_{|H}$ and $\nu_{|H}$ are linear topologies (Proposition 1.20). We are going to prove that $\tau_{|H}$ and $\nu_{|H}$ are compatible.

Let $\chi : (H, \tau) \to \mathbb{T}$ be a continuous character. Since H is an open subgroup of (G, τ), there exists a continuous character $\widetilde{\chi} \in (G, \tau)^\wedge$ extending χ. Since ν is compatible with τ, we obtain $\widetilde{\chi} \in (G, \nu)^\wedge$ and hence $\chi|_H \in (H, \nu|_H)^\wedge$.

Assume conversely, that $\chi \in (H, \nu|_H)^\wedge$. Since χ is $\nu_{|H}$-continuous, there exists an open subgroup $U \in \nu$ such that $\chi(H \cap U) = \{0\}$. Hence the mapping $\chi' : H + U \to \mathbb{T}$, $h + u \mapsto \chi(h)$ ($h \in H$ and $u \in U$) is a well defined homomorphism which extends χ. As $\chi'(U) = \{0\}$, we obtain that χ' is $\nu|_{H+U}$ is continuous. Since $H + U$ is an open subgroup of (G, ν), χ' can be extended to a continuous homomorphism $\widetilde{\chi} \in (G, \nu)^\wedge = (G, \tau)^\wedge$. As above, this shows that $\chi = \widetilde{\chi}|_H \in (H, \tau|_H)^\wedge$.

Since $\tau_{|H}$ is Mackey, we have that $\nu_{|H} \le \tau_{|H}$, and their extensions to G also satisfy $\overline{\nu|_H} \le \overline{\tau|_H}$. From the fact that H is open in τ, we obtain that $\overline{\tau|_H} = \tau$. Thus, $\nu \le \overline{\nu|_H} \le \overline{\tau|_H} = \tau$. Since $\tau_{|H}$ is locally quasi-convex and H a τ-open subgroup, the topology τ is locally quasi-convex and consequently τ is the Mackey topology on G. □

Theorem 3.6 *Let $G = \bigoplus Z_{m_n}$ where (m_n) is a sequence $m_n \to \infty$ and $G = \bigoplus \mathbb{Z}_{m_n}$. Then the metrizable locally quasi-convex topology τ on G inherited from the product $\prod_\omega \mathbb{Z}_{m_n}$ is not Mackey.*

Proof The idea of the proof is to fix appropriately a sequence $\mathbf{c} \subset G^\wedge = \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$, and define the topology $\tau_\mathbf{c}$ of uniform convergence on the range of $\mathbf{c}$. Then we prove that $\tau_\mathbf{c}$ is non-precompact, and strictly finer than τ, but still compatible. Hence, τ is not Mackey.

Observe that the algebraic dual of G is $G^* = \prod \mathbb{Z}_{m_n}^\wedge$ and the topological dual is $G^\wedge = \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$. Clearly, $\mathbb{Z}_{m_n}^\wedge \cong \mathbb{Z}_{m_n}$, and for convenience we make the following identifications:

$$\mathbb{Z}_{m_n}^\wedge = \left(-\frac{m_n}{2}, \frac{m_n}{2}\right] \cap \mathbb{Z}$$

and

$$\mathbb{Z}_{m_n} = \left\{\frac{k}{m_n} : k \in \left(-\frac{m_n}{2}, \frac{m_n}{2}\right] \cap \mathbb{Z}\right\}. \tag{2}$$

For $j_n = \chi_n \in \mathbb{Z}_{m_n}^\wedge$ and $\frac{k_n}{m_n} = x_n \in \mathbb{Z}_{m_n}$ we define $\chi_n(x_n) = j_n \cdot \frac{k_n}{m_n} + \mathbb{Z}$. For $x = (x_1, x_2, \dots) \in G$ and $\chi = (\chi_1, \chi_2, \dots) \in G^\wedge$, we have $\chi(x) = \sum_{n=1}^\infty \chi_n x_n$.

We can assume without loss of generality that the sequence (m_n) is non-decreasing, since a rearrangement of the sequence (m_n) leads to an isomorphic group.

For $n \in \mathbb{N}$, let $(e_n) = (0, \dots, 0, 1, 0, \dots) \in G^\wedge$, with 1 in n-th position and define $\mathbf{c} = \{\pm e_n : n \in \mathbb{N}\} \cup \{0\} \subset G^\wedge$. Identifying $\mathbb{T}$ with $(-\frac{1}{2}, \frac{1}{2}]$, we let $\mathbb{T}_m = \{x \in \mathbb{T} : |x| \le \frac{1}{4m}\}$. Obviously, $\mathbb{T}_m$ is quasi convex in $\mathbb{T}$, so $e_n^{-1}(\mathbb{T}_m)$ is quasi convex in G for each $n \in \mathbb{N}$. Hence,

$$V_m := \{x \in G : e_n(x) \in \mathbb{T}_m \text{ for all } n \in \mathbb{N}\} \overset{e_n(x)=x_n}{=} \left\{x \in G : |x_n| \leq \frac{1}{4m} \text{ for all } n \in \mathbb{N}\right\}$$
$$= \bigcap_{n \in \mathbb{N}} e_n^{-1}(\mathbb{T}_m) \tag{3}$$

is quasi convex as well. Therefore, the family $\{V_m : m \in \mathbb{N}\}$ is a neighborhood basis of 0 for the locally quasi-convex topology $\tau_{\mathbf{c}}$ on G.

Writing the elements $x = (x_n)$ of G, with $x_n = \frac{k_n}{m_n}$ (as in (2)), one has $V_m = \left\{x \in G : |k_n| \leq \frac{m_n}{4m} \text{ for all } n\right\}$.

To conclude the proof of the theorem we need the following lemma:

Lemma 3.7 $(G, \tau_{\mathbf{c}})^\wedge = \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$.

Proof The fact $\bigoplus_\omega \mathbb{Z}_{m_n}^\wedge \leq (G, \tau_{\mathbf{c}})^\wedge$ is straightforward. Indeed, let $\chi \in \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$. Then there exists $n_1 \in \mathbb{N}$ such that $\chi_n = 0$ if $n \geq n_1$. Let $m = m_{n_1}$, hence $x_1 = \cdots = x_{n_1-1} = 0$ for all $x \in V_m$. Then $\chi(V_m) = \{0_{\mathbb{T}}\}$ and χ is continous.

Let now $\chi = (\chi_n) \in \prod_\omega \mathbb{Z}_{m_n}^\wedge \setminus \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$. Then the set $A := \{n \in \mathbb{N} : \chi_n \neq 0\}$ is infinite. Our aim is to prove that $\chi \notin (G, \tau_{\mathbf{c}})^\wedge$. We argue by contradiction. Assume that $\chi \in (G, \tau_{\mathbf{c}})^\wedge$. There exists then $m \in \mathbb{N}$ such that $\chi \in V_m^\triangleright := \{\phi \in (G, \tau_{\mathbf{c}})^\wedge : \phi(V_m) \subset \mathbb{T}_+\}$. Now the proof will be over if we find $x \in V_m$ with $\chi(x) \notin \mathbb{T}_+$.

To this end, we pick

$$n_0 := \min\left\{n \in \mathbb{N} : \frac{m_n}{4m} \geq \frac{5}{2}\right\}. \tag{4}$$

Equivalently, $\dfrac{1}{m_n} \leq \dfrac{1}{10m}$ if $n \geq n_0$.

We describe next in (a) some elements of V_m, which will be suitable for our purposes and in (b) a condition for the components of χ.

Claim 3.8 *(a) If $x_n = 0$ for $n < n_0$ and $x_n \in \{0, \pm\frac{1}{m_n}\}$ for all $n \geq n_0$, then $x = (x_n) \in V_m$.*
(b) $\dfrac{|\chi_n|}{m_n} \leq \dfrac{1}{4}$ for all $n_0 \leq n$.

Proof (a) Indeed, since $\dfrac{1}{m_n} \leq \dfrac{1}{m_{n_0}} \leq \dfrac{1}{10m}$ if $n \geq n_0$, it follows from (3) that $x \in V_m$.

(b) By item (a), we have that $x = (0, \ldots, \overset{(n)}{\frac{1}{m_n}}, \ldots) \in V_m$. Since $\chi \in V_m^\triangleright$, it follows that $|\chi(x)| = \dfrac{|\chi_n|}{m_n} \leq \dfrac{1}{4}$.

□

To continue the proof of Lemma 3.7, define, for each $k \in \left\{0, 1, \ldots, \frac{m_{n_0}}{2}\right\}$, the set

$$A_k := \left\{n \in A : \frac{k}{m_{n_0}} < \frac{|\chi_n|}{m_n} \leq \frac{k+1}{m_{n_0}}\right\}. \tag{5}$$

Then $\left\{A_k : 0 \le k \le \frac{m_{n_0}}{2}\right\}$ is a finite partition of A and at least one A_k is infinite, as A is infinite. In order to define an element $x \in V_m$ which satisfies $\chi(x) \notin \mathbb{T}_+$ we distinguish the cases $k \ge 1$ and $k = 0$.

Case 1: $\mathbf{k} \ge \mathbf{1}$. Since A_k is infinite, one can find $n_0 \le n_1 < n_2 < \cdots < n_{m_{n_0}}$ in A_k. Define

$$x_{n_i} := \frac{1}{m_{n_i}} \operatorname{sign}(\chi_{n_i})$$

By the definition of A_k, we have $\dfrac{k}{m_{n_0}} < \chi_{n_i}(x_{n_i})$ for $1 \le i \le m_{n_0}$. Thus,

$$\sum_{i=1}^{m_{n_0}} \chi_{n_i}(x_{n_i}) > k \ge 1.$$

Taking into account that each summand in the above sum satisfies $\chi_n(x_n) \le \frac{1}{4}$ by Claim 3.8(b), we conclude that there exists a natural $N \le m_{n_0}$ such that

$$\frac{1}{4} < \sum_{i=1}^{N} \chi_{n_i}(x_{n_i}) < \frac{3}{4}.$$

Define

$$x'_n := \begin{cases} \frac{1}{m_{n_i}} \operatorname{sign}(\chi_{n_i}) & \text{when } \ n = n_i \ \text{ for some } \ 1 \le i \le N \\ 0 & \qquad\qquad \text{otherwise} \end{cases}$$

By Claim 3.8(a), $x' = (x'_n) \in V_m$ and by construction $\chi(x') \notin \mathbb{T}_+$.

Case 2: $\mathbf{k} = \mathbf{0}$.

That is: A_0 is infinite. Choose $J \subset A_0$ with $|J| = 2m$. For $n \in J$, define

$$j_n := \frac{\left\lfloor \dfrac{m_n}{4m\,|\chi_n|} \right\rfloor \operatorname{sign}(\chi_n)}{m_n}.$$

First, note that for all $n \in J$ one has $j_n \ne 0$, as $\dfrac{m_n}{4m\,|\chi_n|} \ge \dfrac{m_{n_0}}{4m} > 2$, where the last inequality is due to (4). On the other hand,

$$|j_n| \le \left| \frac{\dfrac{m_n}{4m\,|\chi_n|}}{m_n} \right| = \frac{1}{4m\,|\chi_n|}. \tag{6}$$

This gives $|j_n| \leq \frac{1}{4m}$, as $|\chi_n| \geq 1$. Define

$$x_n := \begin{cases} j_n & \text{when } \; n \in J \\ 0 & \text{otherwise} \end{cases}$$

Then, $x = (x_n) \in V_m$, by Eq. (3).

In addition,

$$|x_n| \geq \frac{\dfrac{m_n}{4m\,|\chi_n|} - 1}{m_n} = \frac{1}{4m\,|\chi_n|} - \frac{1}{m_n}. \tag{7}$$

Combining (6) and (7), we get, for $n \in J$:

$$\frac{3}{20m} = \frac{1}{4m} - \frac{1}{10m} \leq \frac{1}{4m} - \frac{1}{m_{n_0}} \overset{n\in A_0}{\leq} \frac{1}{4m} - \frac{|\chi_n|}{m_n} \overset{(7)}{\leq} |\chi_n|(|x_n|) = \chi_n(x_n) \overset{(6)}{\leq} \frac{1}{4m}.$$

Applying these inequalities to $\chi(x) = \sum_{n\in J} \chi_n x_n$ gives

$$\frac{1}{4} < \frac{3}{10} = \frac{3}{20m}\,|J| \leq \sum_{n\in J} \chi_n x_n = \chi(x) \leq \frac{1}{4m}\,|J| = \frac{1}{2}.$$

This implies that $\chi(x) \notin \mathbb{T}_+$, a contradiction in view of $x \in V_m$.

In both cases this proves that $\chi \notin (G, \tau_{\mathbf{c}})^\wedge$.

Hence $(G, \tau_{\mathbf{c}})^\wedge = \bigoplus_\omega \mathbb{Z}_{m_n}^\wedge$.

In a precompact abelian group the polar of each 0-neighborhood is finite. Since in our case the polar of each neighborhood V_m $(m \in \mathbb{N})$ contains the infinite set $\mathbf{c}$, we conclude that $\tau_{\mathbf{c}}$ is not precompact.

Proof of Theorem C: We have to prove that for any unbounded abelian group G, there exists a metrizable locally quasi-convex group topology which is not Mackey. We divide the proof in three cases, namely:

(a) G is non reduced.
(b) G is non torsion.
(c) G is reduced and torsion.

(a) If G is non reduced, let $d(G)$ be its maximal divisible subgroup. Then $d(G) \cong \bigoplus_\alpha \mathbb{Q} \oplus \bigoplus_{i=1}^\infty \mathbb{Z}(p_i^\infty)^{(\beta_i)}$ [25, Theorem 21.1].

If $\alpha \neq 0$, there exists $H \leq G$, such that $H \cong \mathbb{Q}$. By [8], there exists a metrizable non-Mackey topology on $\mathbb{Q}$ and applying Corollary 3.3, G is not a Mackey group.

If $\alpha = 0$, then there exists p_i satisfying $\mathbb{Z}(p_i^\infty) \leq d(G) \leq G$. By [8], there exists a metrizable, non-Mackey topology on $\mathbb{Z}(p_i^\infty)$. By Corollary 3.3 a metrizable topology on G which is not Mackey can be found.

(b) If G is non torsion, there exists $x \in G$, such that $\langle x \rangle \cong \mathbb{Z}$. Equip $\mathbb{Z}$ with the $\mathbf{b}$-adic topology, which is metrizable but not Mackey [1]. Applying Corollary 3.3, we get the result.

(c) We can write G as the direct sum of its primary components, that is $G = \bigoplus_p G_p$. Consider two cases:

$(c.1)$ $G_p \neq 0$ for infinitely many prime numbers p. Then, for some infinite set Π of prime numbers,we have $\bigoplus_{p\in\Pi} \mathbb{Z}_p \hookrightarrow G$. By Lemma 3.6, there exists a metrizable locally quasi-convex group topology on $\bigoplus_{p\in\Pi} \mathbb{Z}_p$ which is not Mackey. Hence, by Corollary 3.3, there exists a metrizable locally quasi-convex group topology which is not Mackey on G.

$(c.2)$ $G_p \neq 0$ for finitely many p. Then, one of them is unbounded, say G_p. Let $B := \bigoplus_{n=1}^{\infty} \mathbb{Z}_{p^n}^{(\alpha_n)}$ be a basic subgroup of G_p, i.e., B is pure and G_p/B is divisible [25]. Let us recall the fact that purity of B means that $p^n B = p^n G_p \cap B$ for every $n \in \mathbb{N}$.

We aim to prove that B is unbounded. Assume for contradiction that B is bounded, so $p^n B = 0$ for some $n \in \mathbb{N}$. Then $\{0\} = p^n B = p^n G_p \cap B$. On the other hand, $p^n G_p + B = G_p$. Indeed: Fix $n \in \mathbb{N}$ and $x \in G_p$. Since G_p/B is divisible, there exists $g \in G_p$ such that $x + B = p^n(g + B)$ and hence $x = p^n g + b \in p^n G_p + B$ for suitable $b \in B$. Since $p^n G_p \cap B = 0$, we can write $G_p = p^n G_p \oplus B$. Then $p^n G_p \cong G_p/B$ is a divisible subgroup of the reduced group G_p. Hence $p^n G_p \cong G_p/B = 0$ and $G_p = B$ is bounded, a contradiction.

Since B is unbounded there exists a sequence $(n_i) \subseteq \mathbb{N}$ with $\alpha_{n_i} \neq 0$ such that $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{p^{n_i}} \leq B \leq G$. By Theorem 3.6, there exists a metrizable locally quasi-convex group topology on $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{p^{n_i}}$ which is not Mackey. Hence, by Corollary 3.3, there exists a metrizable locally quasi-convex group topology which is not Mackey on G.

Now we can prove the main theorem.

Proof of Theorem A: We have to prove that for an abelian group G the following assertions are equivalent:

(i) Every metrizable locally quasi-convex group topology on G is Mackey.
(ii) G is bounded.

(i) implies (ii) is Theorem C. (ii) implies (i) is Theorem B.

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